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Particles and Interactions

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Towards stabilizing the Complex Langevin method for the real-time evolution of non-Abelian gauge theories

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Introduction to the Complex Langevin method

- **Complex actions lead to a sign problem** $\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathbb{R}} \mathcal{O}(x)\, \exp[-S(x)] dx$

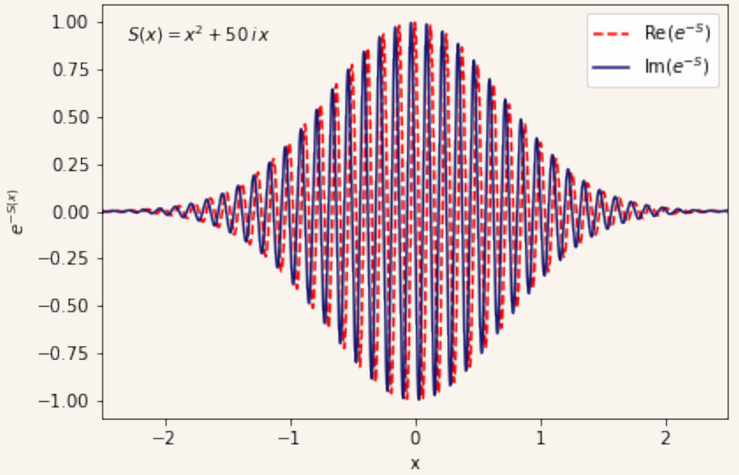
> ***Idea: Complexify the configuration space*** $\mathbb{R} = \mathscr{M} \to \mathscr{M}_c = \mathbb{C}$

- **Complex Langevin equation [1] for stochastic process** $z(\theta) = x(\theta) + iy(\theta)$

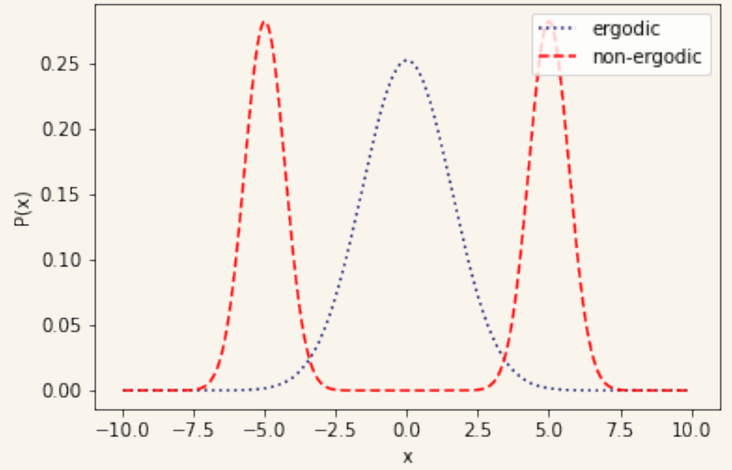
$$\mathrm{Re}\, \dot{z}(\theta) = \mathrm{Re}\, K(z(\theta)) + \eta(\theta),$$
$$\mathrm{Im}\, \dot{z}(\theta) = \mathrm{Im}\, K(z(\theta)),$$

Drift term: $K(z) = -\frac{dS}{dz},\ z \in \mathscr{M}_C = \mathbb{C},$

Noise term: $\langle \eta(\theta) \rangle = 0,\ \langle \eta(\theta)\eta(\theta') \rangle = 2\delta(\theta - \theta')$



- **Complex Fokker-Plank equation**
  - Corresponds to an equation for the probability distribution $P(x, y)$
- **Assumptions for agreement** $\langle \mathcal{O} \rangle_{P(\theta \to \infty)} = \langle \mathcal{O} \rangle_{\rho(\theta \to \infty)}$
  - Ergodicity and sufficient decay of P
  - Regularity of observable $\mathcal{O}$ and drift K



[1] Parisi, G, Wu. Y. (1981). Sci. Sin., 24, 483-496. [inspirehep:9589]

Complex Langevin for lattice Yang-Mills theories

- **Complexification of (gauge) symmetry algebra** $\mathfrak{su}(N) \to \mathfrak{sl}(N,\mathbb{C}) \quad \Rightarrow \quad \mathrm{SU}(N) \to \mathrm{SL}(N,\mathbb{C})$

- **Complex Langevin equation for Yang-Mills (YM) theory**

$$\partial_\theta A^a_\mu(\theta,x) = i\frac{\delta S_{\mathrm{YM}}}{\delta A^a_\mu(\theta,x)} + \eta^a_\mu(\theta,x), \quad \text{YM-action} \quad S_{\mathrm{YM}} = -\frac{1}{4}\int d^4x F^{\mu\nu}_a F^a_{\mu\nu}$$

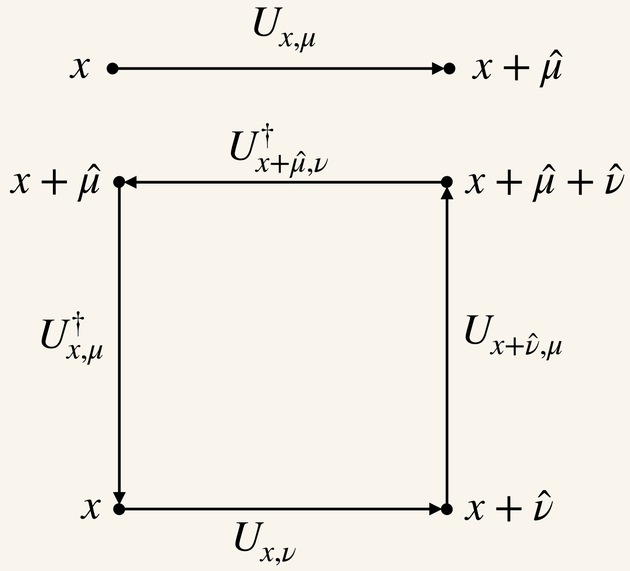
$$\langle \eta^a_\mu(\theta,x)\rangle = 0, \quad \langle \eta^a_\mu(\theta,x)\eta^b_\nu(\theta',y)\rangle = 2\delta(\theta-\theta')\delta^{(d)}(x-y)\delta^{ab}\delta_{\mu\nu}$$

- Introduction of link fields $U_{x,\mu} = \exp\left[ia^\mu \lambda_a A^a_\mu(x + a^\mu/2)\right] \in \mathrm{SU}(N)$

- And so called plaquette variables $U_{x,\mu\nu} = U_{x,\mu}U_{x+\hat{\mu},\nu}U^\dagger_{x+\hat{\nu},\mu}U^\dagger_{x,\nu}$

- Complexifications of gauge fields leads to $U_{x,\mu} \in \mathrm{SL}(N,\mathbb{C})$

- Note that instead of the hermitian adjoint $U^\dagger_{x,\nu}$ we have to invert links $U^{-1}_{x,\nu}$



Discretised Complex Langevin equation for the Wilson action

- **CL update step**

$$U_{x,\mu}(\theta+\epsilon) = \exp\left[it_a(\epsilon K^a_{x,\mu} + \sqrt{\epsilon}\eta^a_{x,\mu})\right] U_{x,\mu}(\theta), \qquad U_{x,\mu} \in \mathrm{SL}(N,\mathbb{C})$$

$$K^a_{x,\mu} = D^a_{x,\mu} S[\{U\}], \quad \langle \eta^a_{x,\nu} \rangle_\eta = 0, \quad \langle \eta^a_{x,\mu}\eta^a_{x,\nu} \rangle_\eta = 2\delta_{\mu\nu}\delta_{xy}\delta_{ab}$$

- **Wilson action**

$$S_W = \frac{1}{N_c}\sum_x \sum_{\mu<\nu} \beta_{\mu\nu} \mathrm{Tr}\left(\mathbf{1} - \frac{1}{2}(U_{x,\mu\nu} + U^{-1}_{x,\mu\nu})\right), \quad U_{x,\mu\nu} = U_{x,\mu}U_{x+\hat{\mu},\nu}U^{-1}_{x+\hat{\nu},\mu}U^{-1}_{x,\nu}$$

$$\beta_{0i} = \beta_{i0} = \frac{2N_c}{g}\frac{a_s}{a_0}, \quad \beta_{ij} = \frac{2N_c}{g}\frac{a_0}{a_s}$$

- **Expectation values via sampling from convergence region**

$$\langle \mathcal{O} \rangle = \frac{1}{Z}\int dU\, \mathcal{O}(U) \exp\left[iS(U)\right] \approx \lim_{\theta_0 \to \infty} \frac{1}{T}\int_{\theta_0}^{\theta_0+T} d\theta\, \mathcal{O}[U(\theta)]$$

Modern stabilization techniques

- **Adaptive step size [2]**

  Counteracts runaways by reducing the step size for large drift terms.

$$\epsilon \mapsto \tilde{\epsilon} = \epsilon \min\left(1, \frac{B}{\max\limits_{x,\mu,a} |K^a_{x,\mu}|}\right)$$

- **Gauge cooling [3]**

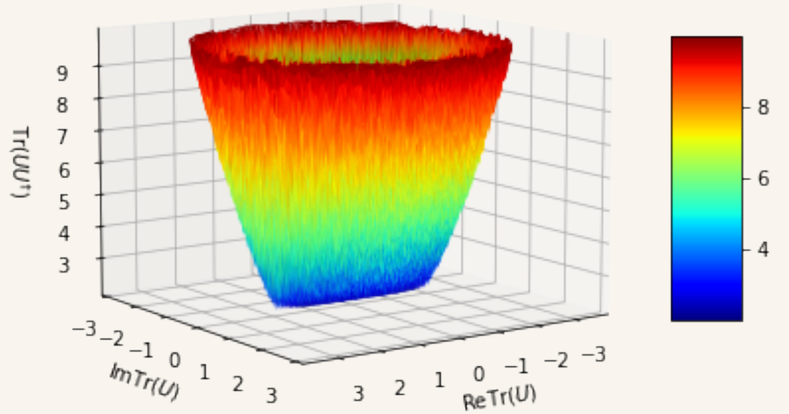
  Reduces the distance to $\mathrm{SU}(N)$. Empirically it was shown, that correct results are obtained for small unitarity norms.

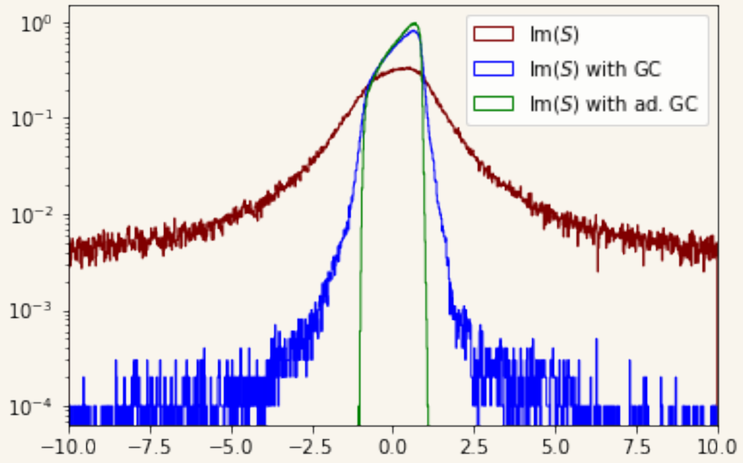
$$U_{x,\mu} \mapsto U^V_{x,\mu} = V_{x,\mu} U_{x,\mu} V^{-1}_{x+\mu,\mu}, \quad F[U] = \sum_{x,\mu} \mathrm{Tr}\left[(U_{x,\mu} U^\dagger_{x,\mu} - 1)^2\right] \to \min$$

- **Dynamical stabilization [4]**

  Penalizes drift terms which lead to excursions into $\mathrm{SL}(N, \mathbb{C})$.

$$K^a_{x,\mu} \mapsto \tilde{K}^a_{x,\mu} = K^a_{x,\mu} + i\alpha M^a_x, \quad M^a_x \propto F[U], \frac{d\mathcal{O}}{d\alpha} \approx 0$$





Polyakov loop model [5]

We introduce an CL-step for anisotropic lattices using a kernel!

[2] Aarts, G. et al. (2010). PLB, 687(2–3), 154–159. [arXiv:0912.3360]

[3] Seiler, E., Sexty, D., Stamatescu, I.-O. (2013). PLB, 723(1–3), 213–216. [arXiv:1211.3709]

[4] Attanasio, F., Jäger, B. (2019). EPJC, 79(1), 16. [arXiv:1808.04400]

[5] Aarts, G. et al. (2013). EPJA, 48, 7. [arXiv:1303.6425]

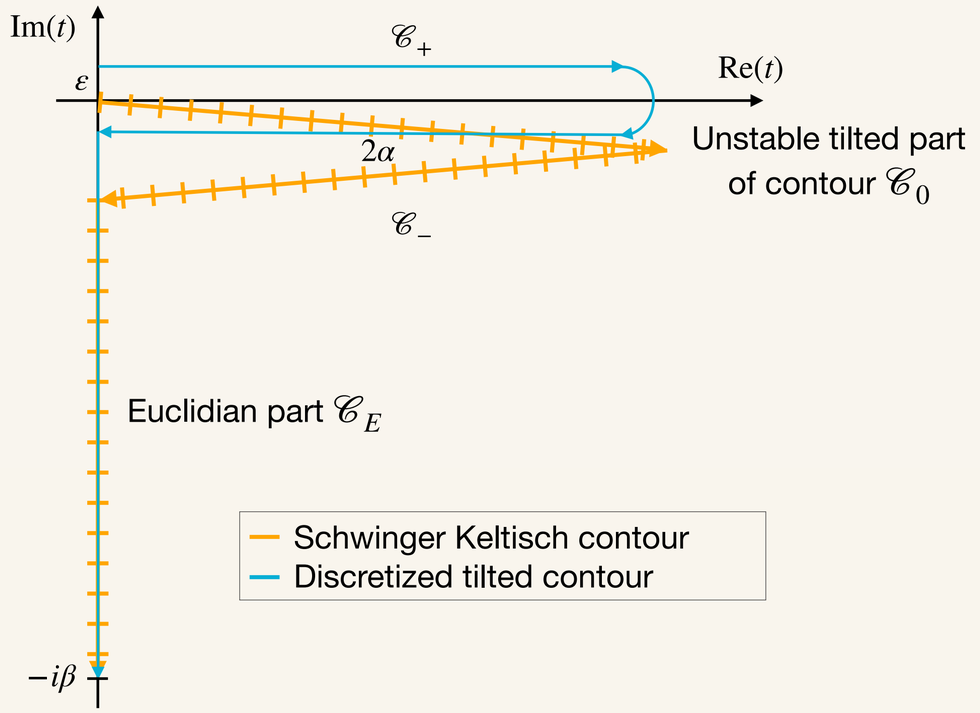
Real-time Yang-Mills simulations in the Schwinger-Keldysh formalism

- Schwinger-Keldysh formalism allows us to express thermal real-time observables

$$\langle O(A)\rangle = \frac{1}{Z}\int \mathscr{D}A_E\, e^{-S_E[A_E]} \int \mathscr{D}'A_+ \mathscr{D}'A_-\, e^{iS[A_+,A_-]}\, O(A)$$

With periodic boundary conditions

$$A^a_\mu(t=0) = A^a_\mu(t=-i\beta)$$



- Non-regularity of discretized path integral with respect to the lattice spacing

  ➔ Regularization by tilting the contour (angle α)

  ➔ Order of limits:
  temporal lattice spacing $a_t \to 0$ then $\alpha \to 0$

- Discretisation of tilted contour gives rise to ambiguities!

  ➔ Complex δ-functions for the noise

  ➔ Complex lattice spacings

Subtleties in the real-time contour discretization

- We parametrize the tilted contour by its arc length (arbitrary but natural choice $d\lambda = |dt|$)
- Real-time complex Langevin equation

$$\partial_\theta A^a_\mu(\theta,\lambda,x^i) = i\frac{\delta S_{\mathrm{YM}}}{\delta A^a_\mu(\theta,\lambda,x^i)} + \eta^a_\mu(\theta,\lambda,x^i), \quad \text{YM-action} \quad S_{\mathrm{YM}} = -\frac{1}{4}\int d\lambda \frac{dt}{d\lambda} d^3x F^{\mu\nu}_a F^a_{\mu\nu}$$

$$\langle \eta^a_\mu(\theta,\lambda,x^i)\rangle = 0, \quad \langle \eta^a_\mu(\theta,\lambda,x^i)\eta^b_\nu(\theta',\lambda',y^i)\rangle = 2\delta(\theta-\theta')\delta(\lambda-\lambda')\delta^{(3)}(x^i-y^i)\delta^{ab}\delta_{\mu\nu}$$

- **Discretized CL update regulates the path integral and has no ambiguities**

$$U_{(\lambda,x^i),\lambda}(\theta+\epsilon) = \exp\left[it^a\left(\tilde{\epsilon}_\lambda\frac{\delta S}{\delta A^a_\lambda} + \sqrt{\tilde{\epsilon}_\lambda}\eta^a_{(\lambda,x^i),\lambda}\right)\right] U_{(\lambda,x^i),\lambda}(\theta), \quad \tilde{\epsilon}_\lambda = \frac{a_\lambda}{a_s}\epsilon$$

$$U_{(\lambda,x^i),j}(\theta+\epsilon) = \exp\left[it^a\left(\tilde{\epsilon}_j\frac{\delta S}{\delta A^a_j} + \sqrt{\tilde{\epsilon}_j}\eta^a_{(\lambda,x^i),j}\right)\right] U_{(\lambda,x^i),j}(\theta), \quad \tilde{\epsilon}_j = \frac{a_s}{\bar{a}_\lambda}\epsilon$$

- a_λ denotes the spacing of the curve parameter λ and $\bar{a}_\lambda = \dfrac{a_\lambda + a_{\lambda-1}}{2}$
- **For constant ϵ and $a_\lambda \to 0$ the noise term of the spatial update diverges**

Introduction of an anisotropic lattice kernel

- Correspondence between Fokker Plank and Langevin equations not unique

  ➔ degrees of freedom can be rescaled independently (kernel freedom)

- We remove the divergence of the spatial noise by rescaling the spatial degrees of freedom by $\bar{a}_\lambda / a_s$
- **Observation: Improved stability by rescaling the temporal link update by $\mathbf{a}_\lambda / \mathbf{a}_\mathbf{s}$** (see next slides)

$$U_{(\lambda,x^i),\lambda}(\theta+\epsilon) = \exp\left[it^a\left(\tilde{\epsilon}_\lambda \frac{\delta S}{\delta A^a_\lambda} + \sqrt{\tilde{\epsilon}_\lambda}\eta^a_{(\lambda,x^i),\lambda}\right)\right] U_{(\lambda,x^i),\lambda}(\theta), \quad \tilde{\epsilon}_\lambda = \left(\frac{a_\lambda}{a_s}\right)^2 \epsilon$$

$$U_{(\lambda,x^i),j}(\theta+\epsilon) = \exp\left[it^a\left(\tilde{\epsilon}_j \frac{\delta S}{\delta A^a_j} + \sqrt{\tilde{\epsilon}_j}\eta^a_{(\lambda,x^i),j}\right)\right] U_{(\lambda,x^i),j}(\theta), \quad \tilde{\epsilon}_j = \epsilon$$

- Note: This rescaling amounts to the kernel freedom (Γ) of complex Langevin

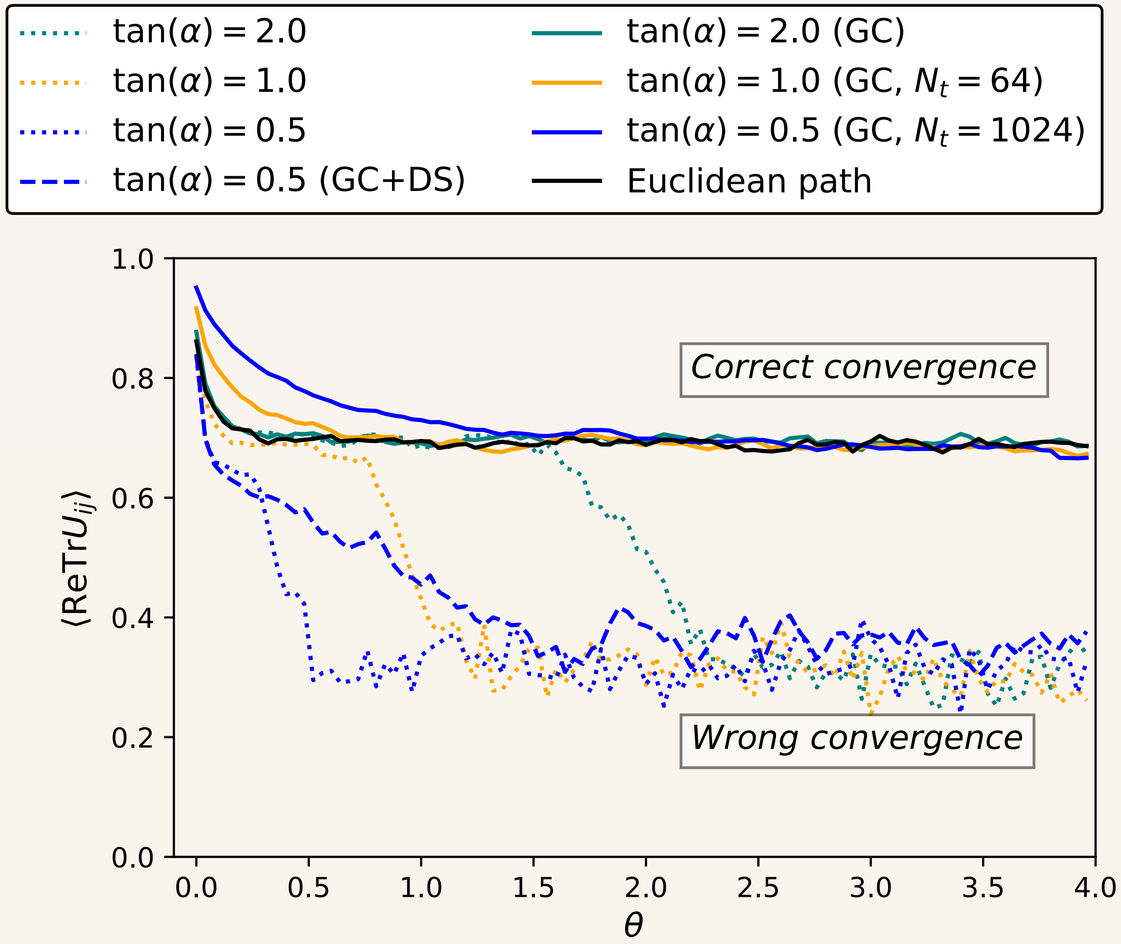
CL-equation: $dA = (\Gamma K + \nabla\Gamma)\, d\theta + \sqrt{\Gamma}\, dw \quad \Longleftrightarrow \quad$ FP-equation: $\partial_\theta P = \nabla\Gamma\,(\nabla - K)\, P$

Thermal expectation values

- Reproduces results of [1] for the average spatial plaquette ($\mathcal{O} = \mathrm{ReTr}U_{ij}$).
- **Stabilisation needed:**

  ➔ short stability region or wrong convergence
- **Time translation invariance** of thermal expectation values

  ➔ enables comparison to results for euclidean path
- DS can improve stability but not always
- Note: Autocorrelation time grows with increasing N_t
- **Partial continuum limit with kernel improves convergence!**



We simulate a $N_t \times 4^3$ lattice with $\beta = 4.0$ and $g = 1$.

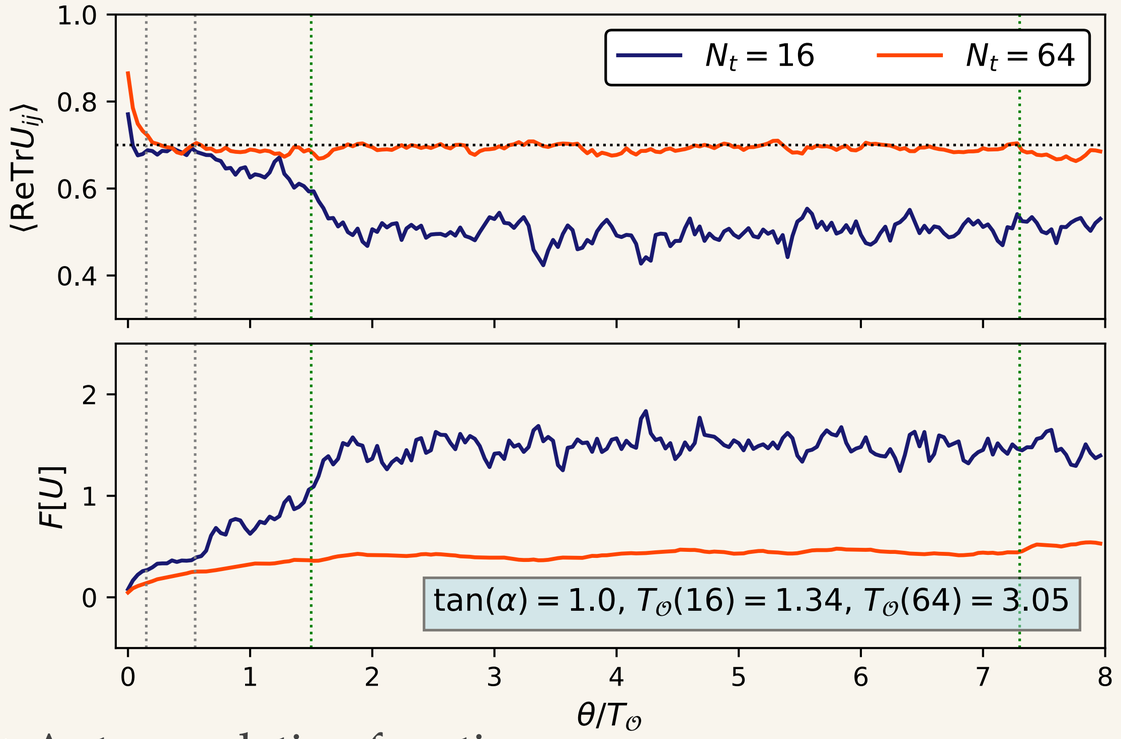
[1] Berges, J. et al. (2007). PRD, 75, 045007. [arXiv:0609058]

Stability region effectively grows

- CL evolution scaled by the autocorrelation time
- Equilibrium density reached also for anisotropic lattice discretization (effectively rescaled θ)
- **Stability region grows faster than autocorrelation length**

  ➔ Novel kernel effectively enlarges stable θ-region
- **No dynamical stabilization is needed**

  ➔ Biased results are avoided



- Autocorrelation function

$$R_{\mathcal{O}}(\tau) = \frac{\langle \left(\mathcal{O}_\theta - \langle \mathcal{O}_\theta \rangle \right) \left(\mathcal{O}_{\theta+\tau} - \langle \mathcal{O}_{\theta+\tau} \rangle \right) \rangle}{\sigma_\theta \sigma_{\theta+\tau}}$$

- Autocorrelation length

$$R_{\mathcal{O}}(\tau) \approx \exp\left(-\tau / T_{\mathcal{O}}\right)$$

Dyson-Schwinger equations

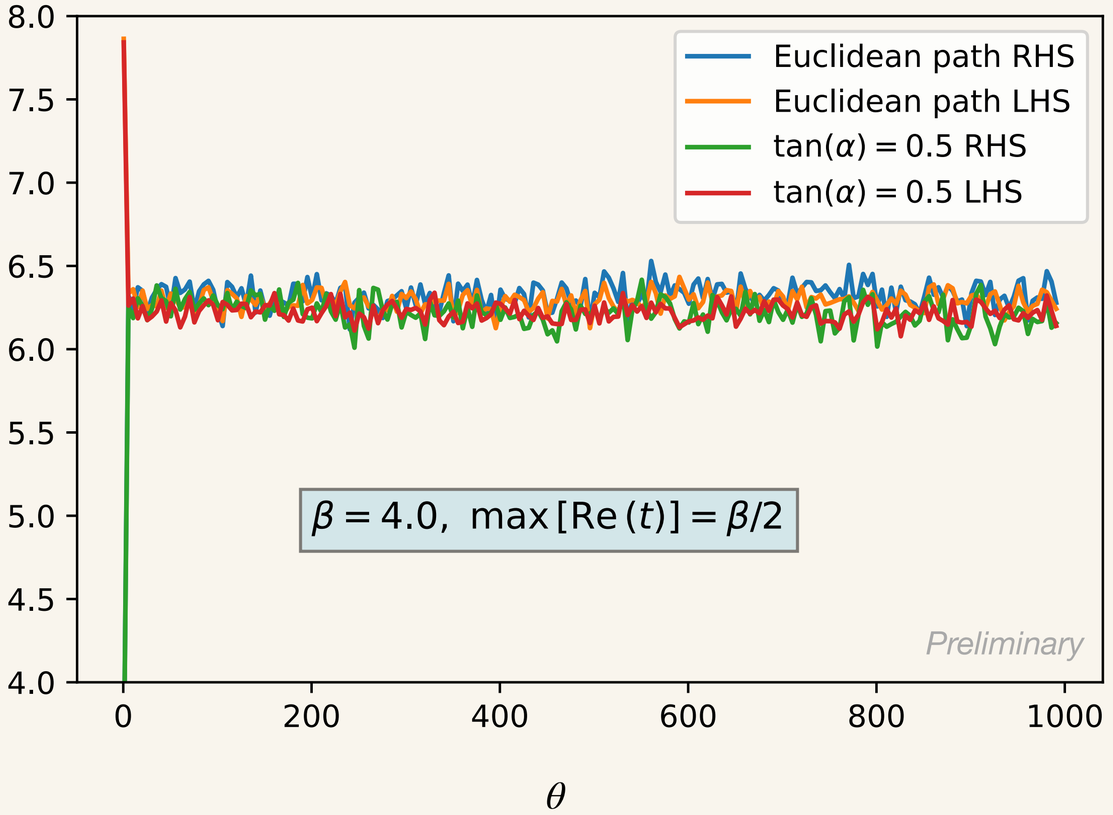
- Dyson-Schwinger equation (DSE) for link fields $U_{x,\mu} \in \mathrm{SL}(N_c, \mathbb{C})$:

$$2\frac{N_c^2-1}{N_c}\langle \mathrm{Tr} U_{x,\mu\nu}\rangle_\eta = \frac{i}{N_c}\sum_{\gamma\neq\mu}\beta_{\mu\gamma}\langle (U^{-1}_{x,\mu\gamma}+U^{-1}_{x,\mu(-\gamma)})U_{x,\mu\nu} - (U_{x,\mu\gamma}+U_{x,\mu(-\gamma)})U_{x,\mu\nu}\rangle_\eta$$

- Figure shows DSE for full tilted SK-contour for $\sum_{i<j} U_{ij}$
- DSEs are derived from the action

  ➔ "First principle" benchmarking observable

- Agreement even for large θ and small tilt angles α



Conclusion & current and future work

Conclusion:

- Stabilization techniques extend the applicability of CL by mitigating known instabilities
- Our introduction of an anisotropic kernel improves stability
- Stability region increases faster with respect to N_t than the autocorrelation time $T_{\mathcal{O}}$
- We checked: DSE, unitarity norm, spatial plaquette

Current and future work:

- Exploration of our approach: can we reach the continuum Schwinger-Keldysh contour?
- Compute unequal-time correlation functions (Wightman functions, spectral functions)
- Technical improvements: boundary terms, adaptive dynamical stabilization, higher order solvers

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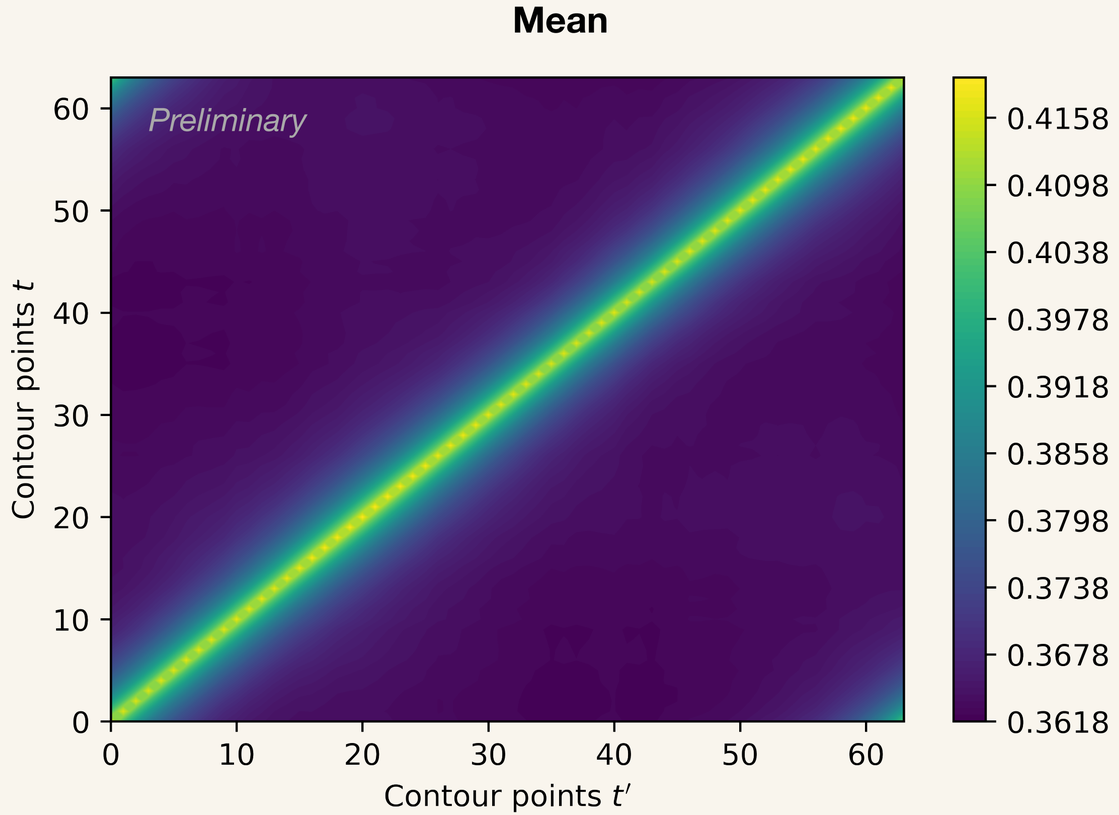
Back up slides

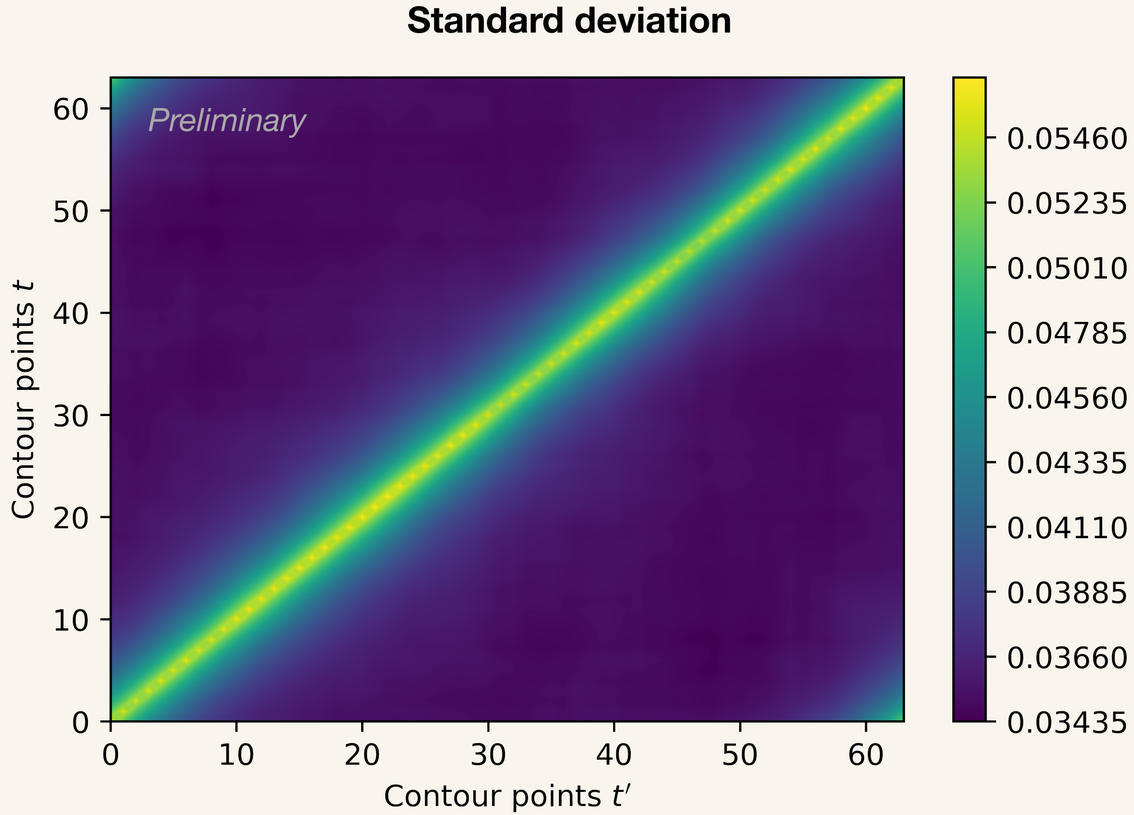
Unequal time correlation functions for Euclidean path

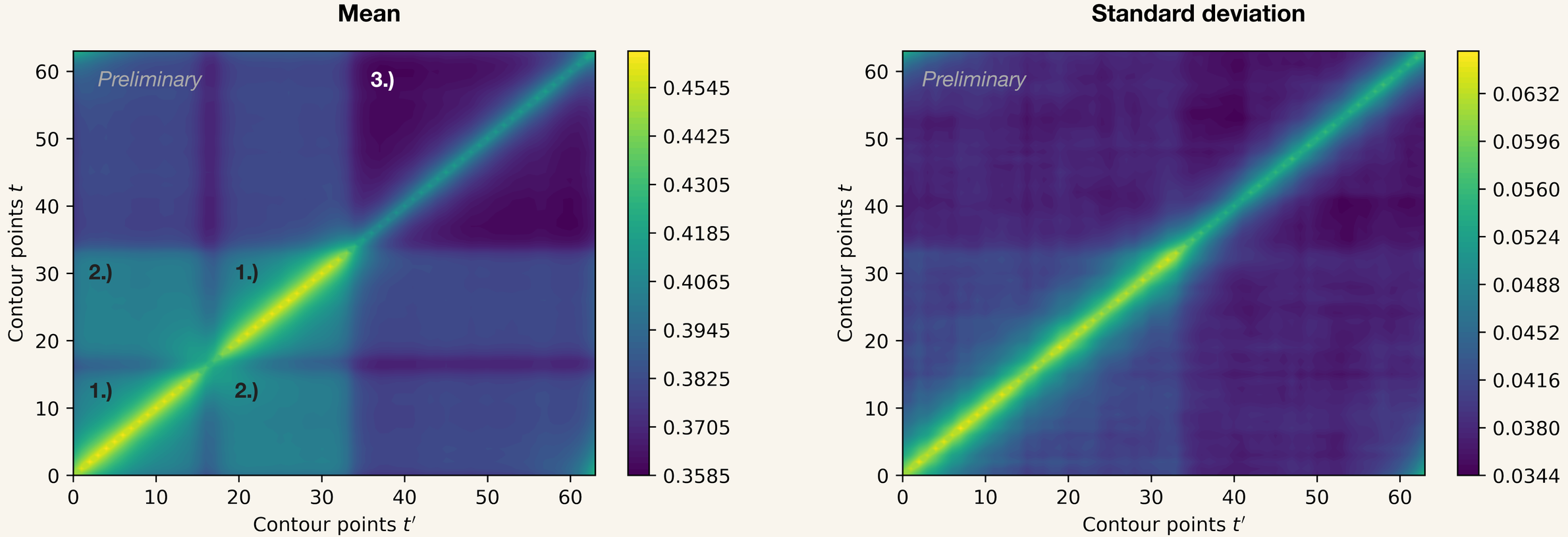
- We look at the centered average spatial plaquette

$$\mathcal{O}(t) = N_c - \frac{N_c}{4}\left[U_{x,ij} + U_{x,j(-i)} + U_{x,(-i)(-j))} + U_{x,(-i)j}\right]$$

- Unequal time correlation function $\langle \mathcal{O}(t)\mathcal{O}(t') \rangle$





Unequal time correlation functions for tilted Schwinger-Keldysh path

- For tilted contour sectors of contour plot correspond to
  1. Time- and anti-timeordered Feynman propagator $D^F, D^{\bar{F}}$
  2. Forward and backward Wightman function $D^{>}(t,t') = \langle\phi(t')\phi(t)\rangle$, $D^{<}(t,t') = \langle\phi(t)\phi(t')\rangle$
  3. Euclidean correlation function